2. DESARGUES' THEOREM

§2.1. Orthogonality

Girard Desargues
1591-1661

Recall that if $\mathbf{a}$, $\mathbf{b}$ $\in$in F^3 are defined to be **orthogonal** if $\mathbf{a}.\mathbf{b} = 0$. Recall, too, that the orthogonal complement of a subspace U is:

$\mathbf{U}^\perp = \{\mathbf{v} \mid \mathbf{u}.\mathbf{v} = 0 \text{ for all } \mathbf{u} \in U\}$.

In other words $U^\perp$ is the set of all vectors that are orthogonal to every vector in U. I showed that $U \leftrightarrow U^\perp$ is a 1-1 correspondence between the projective points and the projective lines. This means that, just as we can represent any projective point as $\langle\mathbf{p}\rangle$ using one vector, so any projective line can be represented in the form $\langle\mathbf{q}\rangle^\perp$.

In the case of $F = \mathbb{R}$, you can think of $\langle\mathbf{p}\rangle$ as the line through the origin that passes through $\mathbf{p}$ and $\langle\mathbf{q}\rangle^\perp$ as the plane through the origin that is perpendicular to $\mathbf{q}$. So the projective point $\langle\mathbf{p}\rangle$ lies on the projective line $\langle\mathbf{q}\rangle^\perp$ if and only if $\mathbf{p}$ and $\mathbf{q}$ are orthogonal.

Theorem 1: If $h = \langle\mathbf{a}\rangle^\perp$ and $k = \langle\mathbf{b}\rangle^\perp$ are distinct projective lines then $h \cap k = \langle\mathbf{a} \times \mathbf{b}\rangle$.

Proof: Since $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$,

$\langle \mathbf{a} \times \mathbf{b} \rangle \le \langle \mathbf{a} \rangle^{\perp} \cap \langle \mathbf{b} \rangle^{\perp}$. But both of these subspaces are projective points and so have dimension 1, which means they must be equal.

Example 1:
Does $\langle(1, -3, 2)\rangle$ lie on the line $\langle(-11, 1, 7)\rangle^{\perp}$?
Solution: Yes, since:

$$(1, -3, 2).(-11, 1, 7) = -11 - 3 + 14 = 0.$$

Example 2: If $A = \langle(1, 3, 2)\rangle$ and $B = \langle(5, 1, 9)\rangle$ find the line AB in the form $\langle \mathbf{p} \rangle^{\perp}$.
Solution: $AB = \langle(1, 3, 2), (5, 1, 9)\rangle = \langle \mathbf{p} \rangle^{\perp}$ where **p** is a non-zero vector orthogonal to both (1, 3, 2) and (5, 1, 9). Clearly such a vector is the cross-product.

$$(1, 3, 2) \times (5, 1, 9) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 2 \\ 5 & 1 & 9 \end{vmatrix} = (25, 1, -14) \text{ so:}$$

$$AB = \langle(25, 1, -14)\rangle^{\perp}.$$

Example 3:
If $h = \langle(1, 3, 2)\rangle^{\perp}$ and $k = \langle(5, 1, 9)\rangle^{\perp}$ find $h \cap k$.
Solution: We need to find a vector that is orthogonal to both vectors (1, 3, 2) and (5, 1, 9). Once again we may take the cross-product $(1, 3, 2) \times (5, 1, 9) = (25, 1, -14)$. So $h \cap k = \langle(25, 1, -14)\rangle$.

Example 4: Suppose $A = \langle(1, 3, 5)\rangle$, $B = \langle(1, -1, 1)\rangle$ and $h = \langle(2, 1, 0)\rangle^{\perp}$. Find $AB \cap h$.

Solution: $(1, 3, 5) \times (1, -1, 1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 5 \\ 1 & -1 & 1 \end{vmatrix} = (8, 4, -4)$ so $AB = \langle(8, 4, -4)\rangle^\perp = \langle(2, 1, -1)\rangle^\perp$.

$$(2, 1, -1) \times (2, 1, 0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 2 & 1 & 0 \end{vmatrix} = (1, 2, 0) \text{ so:}$$

$$AB \cap h = \langle(1, 2, 0)\rangle.$$

The following table translates properties of projective points and lines into the concepts of linear algebra and back again. In proving theorems of projective geometry by linear algebra techniques we convert our assumptions into the language of linear algebra, work with them using the standard techniques of linear algebra and then convert our conclusions back to the language of projective geometry.

Let $P = \langle\mathbf{p}\rangle$, $Q = \langle\mathbf{q}\rangle$, $R = \langle\mathbf{r}\rangle$ be projective points and $a = \langle\mathbf{a}\rangle^\perp$, $b = \langle\mathbf{b}\rangle^\perp$, $C = \langle\mathbf{c}\rangle^\perp$ be projective lines (where all these vectors are non-zero).

PROJ GEOMETRY	**LINEAR ALGEBRA**
projective point	1-dimensional subspace
projective line	2-dimensional subspace
P = Q	$\mathbf{p} = \lambda\mathbf{q}$ for some real $\lambda \neq 0$
P lies on *a* or *a* passes through P	$\mathbf{p}.\mathbf{a} = 0$
P, Q, R are collinear	**p**, **q**, **r** are linearly dependent
a, *b*, *c* are concurrent	**a**, **b**, **c** are linearly dependent

projective line PQ	2-dimensional subspace $P + Q = \langle \mathbf{p} \times \mathbf{q} \rangle$
intersection of projective lines a, b	1-dimensional subspace $a \cap b = \langle \mathbf{a} \times \mathbf{b} \rangle$

§2.2. The Collinearity Lemma

If $P = \langle (x, y, z) \rangle$ is a projective point, we say that x, y, z are **homogeneous coordinates** for P.

Of course they're not unique. For example $\langle (1, 2, 3) \rangle = \langle (2, 4, 6) \rangle$. This fact is exploited by the following lemma which provides a particularly simple set of homogeneous coordinates for three or four collinear points. I call this the **Collinearity Lemma**.

Theorem 2 (COOPER):
If $P = \langle \mathbf{p} \rangle$, Q, R, S are collinear projective points such that P, Q, R are distinct and $P \neq S$, then for a suitably chosen vector $\mathbf{q}$ and scalar λ we may express the four points as:
$P = \langle \mathbf{p} \rangle$, $Q = \langle \mathbf{q} \rangle$, $R = \langle \mathbf{p} + \mathbf{q} \rangle$, $S = \langle \lambda\mathbf{p} + \mathbf{q} \rangle$.
Proof: We break the proof into 12 separate steps.
(1) Since $P = \langle \mathbf{p} \rangle$ has dimension 1, $\mathbf{p} \neq \mathbf{0}$.
(2) Since P lies on $QR = Q + R$ we may write

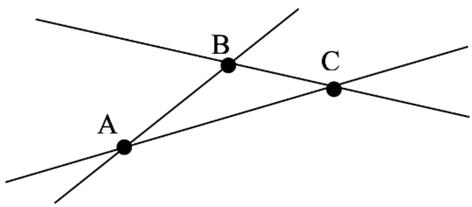
$$\mathbf{p} = \mathbf{q}_1 + \mathbf{r} \text{ for some } \mathbf{q}_1 \in Q, \mathbf{r} \in R.$$

(3) Now if $\mathbf{r} = \mathbf{0}$ we would have $\mathbf{p} = \mathbf{q}_1$ and so $P = Q$, a contradiction. Hence $\mathbf{r} \neq \mathbf{0}$.
(4) Thus $R = \langle \mathbf{r} \rangle$.
(5) Now $\mathbf{r} = \mathbf{p} - \mathbf{q}_1 = \mathbf{p} + \mathbf{q}$ if we define $\mathbf{q} = -\mathbf{q}_1$.

(6) If $\mathbf{q} = \mathbf{0}$ then $\mathbf{r} = \mathbf{p}$ and so R = P, a contradiction. Hence $\mathbf{q} \neq 0$.
(7) Since any non-zero vector in a 1-dimensional subspace spans that subspace we must have $Q = \langle \mathbf{q} \rangle$ and $R = \langle \mathbf{r} \rangle = \langle \mathbf{p} + \mathbf{q} \rangle$.
(8) Since Q lies on PS, $\mathbf{q} \in P + S = \langle \mathbf{p} \rangle + S$.
(9) Thus $\mathbf{q} = \lambda_1 \mathbf{p} + \mathbf{s}$ for some $\lambda_1 \in \mathbb{R}$ and $\mathbf{s} \in S$.
(10) Now $\mathbf{s} = (-\lambda_1)\mathbf{p} + \mathbf{q} = \lambda \mathbf{p} + \mathbf{q}$ if we define $\lambda = -\lambda_1$.
(11) If $\mathbf{s} = \mathbf{0}$ then $\mathbf{q} = \lambda_1 \mathbf{p}$ and so Q = P, a contradiction. Hence $\mathbf{s} \neq \mathbf{0}$.
(12) Therefore $S = \langle \mathbf{s} \rangle = \langle \lambda \mathbf{p} + \mathbf{q} \rangle$.

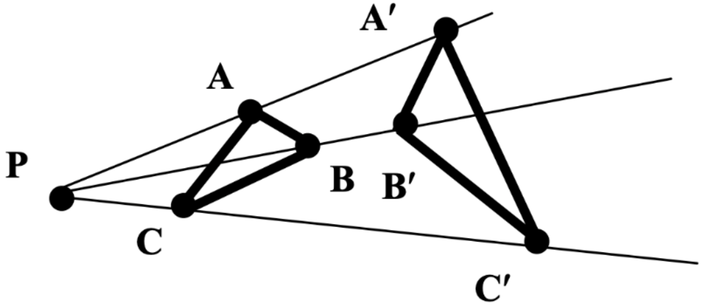
§2.3. Perspective Triangles

A **triangle** ΔABC is a set of distinct projective points A, B, C together with the projective lines AB, AC, BC.

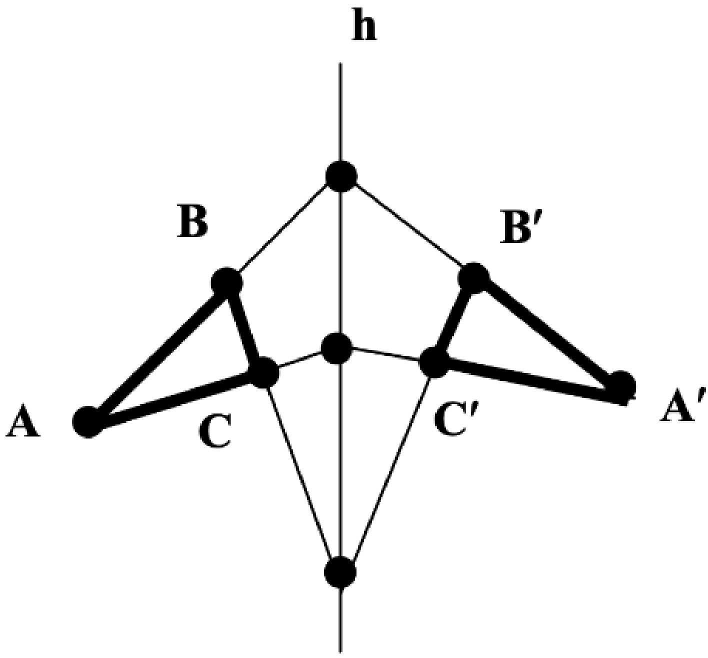


ΔABC is said to be **in perspective with** ΔA′B′C′ **from the point** P if:

{P, A, A′}, {P, B, B′} and {P, C, C′} are three sets of collinear points.



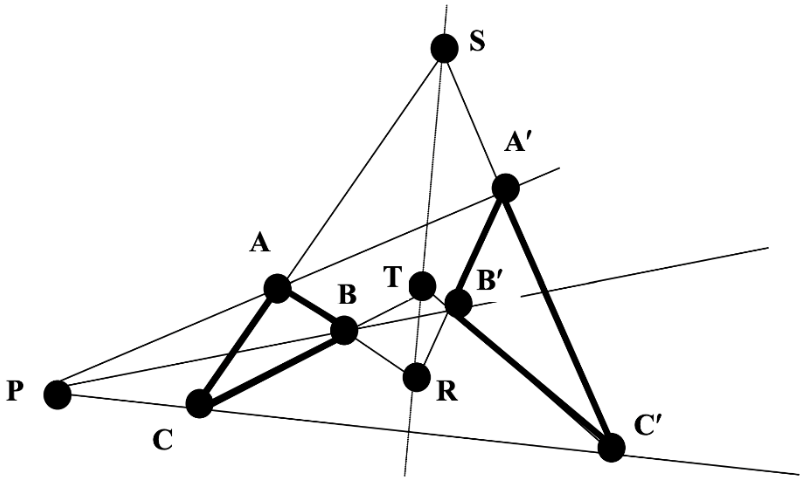
ΔABC is said to be **in perspective with** ΔA′B′C′ **from the line** *h* if {*h*, AB, A′B ′}, {*h*, AC, A′C′} and {*h*, BC, B′C′} are three sets of concurrent lines.



§2.4. Desargues' Theorem

Girard Desargues (1591–1661), a French architect and mathematician who lived in Lyons and Paris, was one of the founders of projective geometry. In 1639 he introduced many of the basic concepts. His proofs didn't use linear algebra (which wasn't developed until the 19th century) and are rather more complicated than the ones presented here. The basic tool for several of these theorems is the Collinearity Lemma. (Theorem 2).

Theorem 3: (DESARGUES) Two triangles are in perspective from a point if and only if they are in perspective from a line.



Proof:

Suppose {P, A, A′}, {P, B, B′} and {P, C, C′} are three collinear sets of points on distinct lines. (The result is trivial if we allow two of the lines to coincide.)

Let R = AB∩A′B′, S = AC∩A′C′ and T = BC∩B′C′.

We must show that R, S, T are collinear.
By the collinearity lemma we may write, for suitable vectors **p**, **a**, **b**, **c**:

$$P = \langle \mathbf{p} \rangle,\ A = \langle \mathbf{a} \rangle,\ A' = \langle \mathbf{p} + \mathbf{a} \rangle,$$
$$B = \langle \mathbf{b} \rangle,\ B' = \langle \mathbf{p} + \mathbf{b} \rangle,$$
$$C = \langle \mathbf{c} \rangle,\ C' = \langle \mathbf{p} + \mathbf{c} \rangle$$

Now $\mathbf{a} - \mathbf{b} \in A + B$ and $\mathbf{a} - \mathbf{b} = (\mathbf{p} + \mathbf{a}) - (\mathbf{p} + \mathbf{b}) \in A' + B'$. Since $\mathbf{a} \neq \mathbf{b}$, $\mathbf{a} - \mathbf{b}$ is a non-zero vector.
Let $R = \langle \mathbf{a} - \mathbf{b} \rangle$. Since R is a subspace of A + B, as a projective point R lies on the projective line AB.

Similarly R lies on A′B′ and so $R = \langle \mathbf{a} - \mathbf{b} \rangle = AB \cap A'B'$.
By defining $S = \langle \mathbf{a} - \mathbf{c} \rangle$ and $T = \langle \mathbf{b} - \mathbf{c} \rangle$ we have:

$$S = AC \cap A'C' \text{ and}$$
$$T = = BC \cap B'C'.$$

Now $(\mathbf{a} - \mathbf{b}) + (\mathbf{b} - \mathbf{c}) + (\mathbf{c} - \mathbf{a}) = \mathbf{0}$ so $\mathbf{a} - \mathbf{b}$, $\mathbf{b} - \mathbf{c}$ and $\mathbf{c} - \mathbf{a}$ are linearly dependent. Hence R, S, T are collinear.

This proves only one half of the theorem, namely that if two triangles are in perspective from a point they are in perspective from a line. We ought now to prove the converse. But once we’ve developed the Principle of Duality the converse will come for free.

§2.5. Duality

If P is a projective point (1-dimensional subspace) then $P^\perp$ is a projective line (2-dimensional subspace) and if h is a projective line then $h^\perp$ is a projective point. The relation $P \leftrightarrow P^\perp$ (or equivalently $h \leftrightarrow h^\perp$) establishes a 1-1 correspondence between the points and lines of the Real Projective Plane. Moreover it interacts with the incidence structure in a very nice way.

Theorem 1: The projective point P lies on the projective line h if and only if the projective point $h^\perp$ lies on the projective line $P^\perp$.

Proof: Let $P = \langle \mathbf{p} \rangle$ and let $h = \langle \mathbf{v} \rangle^\perp$. Then P lying on h is equivalent to **p** and **v** being orthogonal.

But $h^\perp = \langle \mathbf{v} \rangle^{\perp\perp} = \langle \mathbf{v} \rangle$ and so $h^\perp$ lying on $P^\perp$ is also equivalent to **p** and **v** being orthogonal. So the two geometric statements are equivalent to one another.

As a consequence of this very innocent-looking theorem we can establish the following very powerful Principle of Duality.

The Principle of Duality

Any theorem in projective geometry that can be expressed in terms of the following six concepts remains true if these concepts are interchanged as follows:

projective point $\leftrightarrow$ *projective line*

lies on $\leftrightarrow$ *passes through*
collinear $\leftrightarrow$ *concurrent*

A concept or theorem that is obtained by the above interchanges is called the **dual** of the original one. What this principle means is that every time we prove a theorem in projective geometry we have automatically proved another theorem – its dual. Well that's not quite true because sometimes a theorem is its own dual. In the case of Desargues' the dual of what we proved is the converse, which is why we didn't bother with proving the converse.

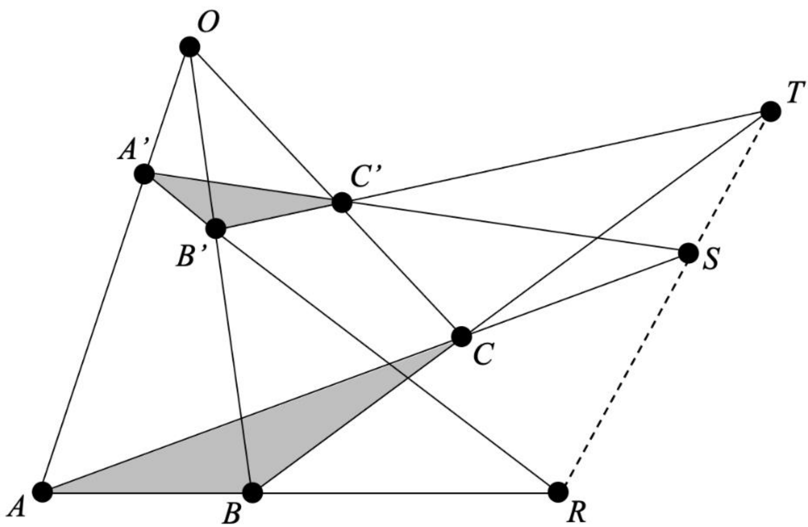
The dual of the property of two triangles being in perspective from a point is the property of two triangles being in perspective from a line. (It's not just because we've changed the word point to line. You need to look at their definitions to see that the definitions are duals of one another.) We proved that if triangles are in perspective from a point then they're in perspective from a line. The dual, which must be true by the principle of duality, is that if triangles are in perspective from a line then they're in perspective from a point.

Sometimes, as in Desargues' theorem, the dual turns out to be the converse. Sometimes it turns out to be the same theorem (a self-dual theorem). Sometimes it's a totally different theorem.

§2.6. Euclidean Interpretation of Desargues' Theorem

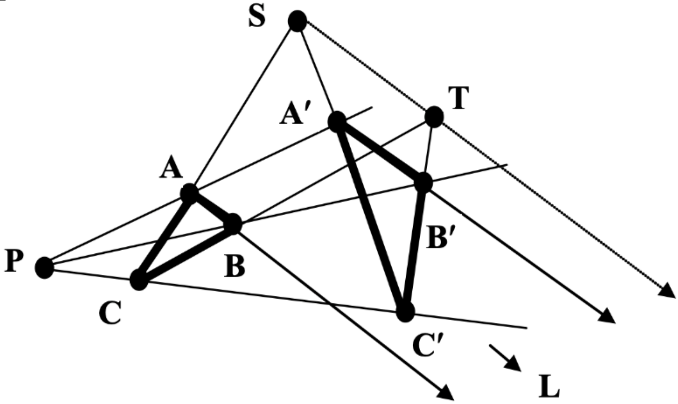
Interpretation 1: All points ordinary

Since Desargues' Theorem is true for the Real Projective Plane it must hold for the Real Affine Plane inside it, that is, ordinary points and ordinary lines relative to some embedding of the Real Affine Plane in the Real Projective Plane. This is the case illustrated by the following diagram.



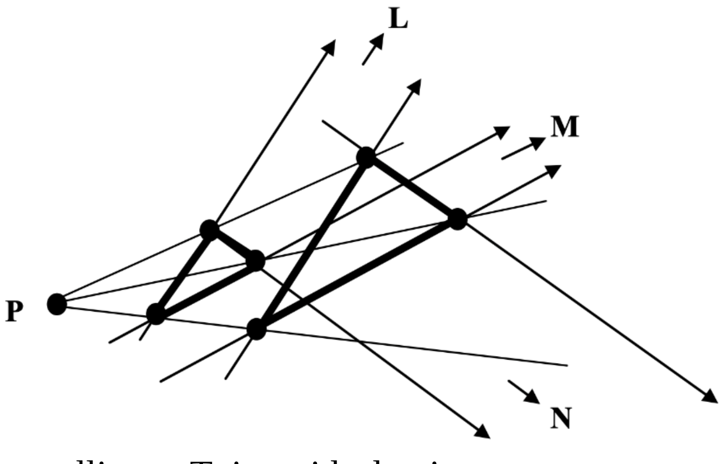
However if some of the points are taken to be ideal points we obtain different affine interpretations of the same projective theorem. If we had attempted to prove them just for the affine plane we'd need separate proofs for each of them.

Interpretation 2: R is ideal



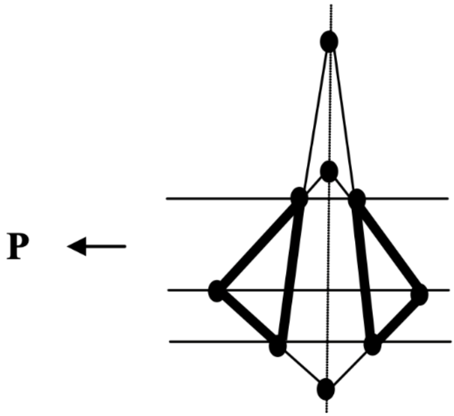
If AB || A′B′ and S = AC$\cap$A′C′ and T = BC$\cap$B′C′ then ST || AB.

Interpretation 2: Both R, S ideal



Since R, S, T are collinear T is an ideal point.
So if AB || A′B′ and AC || A′C′ then BC || B′C′.

Interpretation 3: P is ideal



If AA′, BB′ and CC′ are parallel then R = AB∩A′B′, S = AC∩A′C′ and T = BC∩B′C′ are collinear.

The great power of Projective Geometry is illustrated here. Quite apart from the fact that algebraic methods are simpler than geometric ones (and what could be simpler than (**a** – **b**) + (**b** – **c**) + (**c** – **a**) = **0?**) we can prove several affine theorems by taking suitable interpretations of a single projective theorem.

But wait! There's more! The Principle of Duality means that every time we prove a projective theorem we get a second one for free. And this, too, will have several affine interpretations.

EXERCISES FOR CHAPTER 2

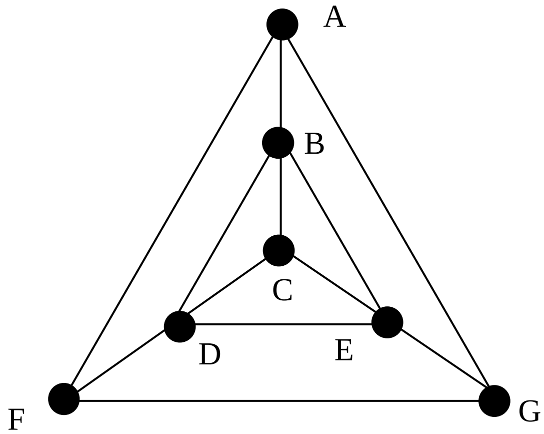
Exercise 1: Let A = ⟨(1, 2, 4)⟩, B = ⟨(5, –3, 2)⟩, C = ⟨(–3, 7, 6)⟩ and D = ⟨(13, –13, –2)⟩.
(a) Find AB and show that C, D both lie on AB.
(b) Use the proof of the Collinearity Lemma to find vectors **a**, **b** and a scalar λ such that:

A = ⟨**a**⟩, B = ⟨**b**⟩, C = ⟨**a** + **b**⟩ and D = ⟨λ**a** + **b**⟩.

Exercise 2: If P = ⟨(1, 2, 8)⟩, Q = ⟨(4, 1, 5)⟩, R = ⟨(–5, 4, 14)⟩, S = ⟨(14, 7, 31)⟩, find vectors **p**, **q** and a scalar λ such that:

P = ⟨**p**⟩, Q = ⟨**q**⟩, R = ⟨**p** + **q**⟩, S = ⟨λ**p** + **q**⟩.

Exercise 3:
(a) In the following diagram AF is parallel to BD and AG is parallel to BE. Find two triangles that are in perspective from a point.

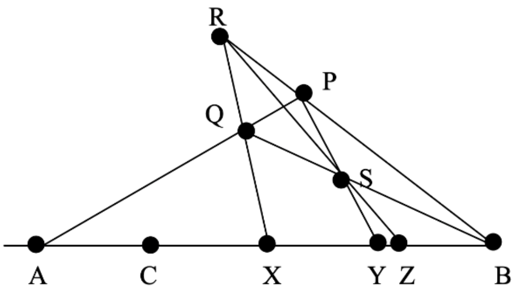


(b) State the Euclidean interpretation of Desargues' Theorem for this configuration.

Exercise 4:
Let $A = \langle(1, 0, 0)\rangle$, $B = \langle(0, 1, 0)\rangle$, $C = \langle(0, 0, 1)\rangle$,
$A' = \langle(2, 1, 1)\rangle$, $B' = \langle(2, 3, 2)\rangle$, $C' = \langle(3, 3, 4)\rangle$.
Let $R = AB \cap A'B'$, $S = AC \cap A'C'$ and $T = BC \cap B'C'$.
Find R, S and T and verify that they are collinear.

Exercise 5: In each of the following cases find a point P and a line h such that the triangles ABC and $A'B'C'$ are in perspective from the point P and from the line h.
(i) $A = (0, 2)$, $B = (1, 1)$, $C = (1, 0)$,
$A' = (1, 2)$, $B' = (2, 1)$, $C' = (0, 0)$.
These points are in the Euclidean plane embedded in the real projective plane.
(ii) $A = \langle(1, 4, 7)\rangle$, $B = \langle(2, -1, 3)\rangle$, $C = \langle(3, 5, 3)\rangle$,
$A' = \langle(2, 3, 5)\rangle$, $B' = \langle(1, 1, -1)\rangle$, $C' = \langle(1, -4, -2)\rangle$.

Exercise 6: In the following diagram A, B, P are not collinear while ACX, YZB, AQP, RQX, RSZ, RPB, QSB are straight lines.



Let

$A = \langle \mathbf{a} \rangle$, $B = \langle \mathbf{b} \rangle$, $C = \langle \mathbf{a} + \mathbf{b} \rangle$

$X = \langle \mathbf{a} + x\mathbf{b} \rangle$, $Y = \langle \mathbf{a} + y\mathbf{b} \rangle$, $Z = \langle \mathbf{a} + z\mathbf{b} \rangle$,

$P = \langle \mathbf{p} \rangle$, $Q = \langle \mathbf{a} + \mathbf{p} \rangle$.

(i) Prove that $R = \langle \mathbf{p} - x\mathbf{b} \rangle$;

(ii) Find S in terms of **a**, **b**, **p**, x, y, z;

(iii) Use the fact that RSZ are collinear to prove that

$$x + y = z.$$

Exercise 7:

Let A, B, C and D be four distinct points, no three of which are collinear. Let S and R be points on AB and AD respectively, and let $K = AC \cap BD$, $M = KS \cap BC$ and $N = KR \cap CD$.

Prove that the lines BD, MN and RS are concurrent.

SOLUTIONS FOR CHAPTER 2

Exercise 1:

$AB = \langle (1, 2, 4) \times (5, -3, 2) \rangle^{\perp} = \langle (16, 18, -13) \rangle^{\perp}$.

Since $(-3, 7, 6).(16, 18, -13) = 0$ and

$(13, -13, -2).(16, 18, -13) = 0$,

both C and D lie on AB.

Suppose $(-3, 7, 6) = x(1, 2, 4) + y(5, -3, 2)$.

Then $\begin{cases} x + 5y = -3 \\ 2x - 3y = 7 \\ 4x + 2y = 6 \end{cases}$.

Solving, we get $x = 2$, $y = -1$.

Hence (–3, 7, 6) = 2(1, 2, 4) – (5, –3, 2).
Therefore we let **a** = 2(1, 2, 4) = (2, 4, 8) and
b = –(5, –3, 2) = (–5, 3,–2).
Then A = ⟨**a**⟩, B = ⟨**b**⟩ and C = ⟨**a** + **b**⟩.
Suppose (13, –13, 2) = x(1, 2, 4) + y(5, –3, 2).

$$\text{Then } \begin{cases} x + 5y = 13 \\ 2x - 3y = -13 \\ 4x + 2y = 2 \end{cases}.$$

Solving, we get $x = -2$, $y = 3$.
Hence (–3, 7, 6) = –2(1, 2, 4) + 3(5, –3, 2) = –**a** – 3**b**.
Let **c** = (–1/3)(–3, 7, 6). Then c = (1/3)**a** + **b**.
Hence **a** = (2, 4, 8), **b** = (–5, 3, –2) and λ = 1/3.

Exercise 2:
Let x(1, 2, 8) + y(4, 1, 5) + z(–5, 4, 14) = (0, 0, 0).

$$\begin{pmatrix} 1 & 4 & -5 \\ 2 & 1 & 4 \\ 8 & 5 & 14 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & -5 \\ 0 & -7 & 14 \\ 0 & -27 & 54 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & -5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $z = 1$. ∴ $y = 2$, $x = -3$.
∴ –3(1, 2, 8) + 2(4, 1, 5) + (–5, 4, 14) = (0, 0, 0).
∴ (–5, 4, 14) = 3(1, 2, 8) – 2(4, 1, 5).
Let **p** = (3, 6, 24) and **q** = (–8, –2, –10).
∴ P = ⟨**p**⟩, Q = ⟨**q**⟩, R = ⟨**p** + **q**⟩.
Let x(1, 2, 8) + y(4, 1, 5) + z(14, 7, 31) = (0, 0, 0).

$$\begin{pmatrix} 1 & 4 & 14 \\ 2 & 1 & 7 \\ 8 & 5 & 31 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 14 \\ 0 & -7 & -21 \\ 0 & -27 & -81 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 14 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $z = 1$. ∴ $y = -3$, $x = -2$.

$\therefore$ (14, 7, 31) = 2(1, 2, 8) + 3(4, 1, 5) = (2/3)**p** – (3/2)**q**
$\therefore$ –(3/2)(14, 7, 31) = (–4/9)**p** + **q**.
So S = $\langle\lambda$**p** + **q**$\rangle$ where λ = –4/9.

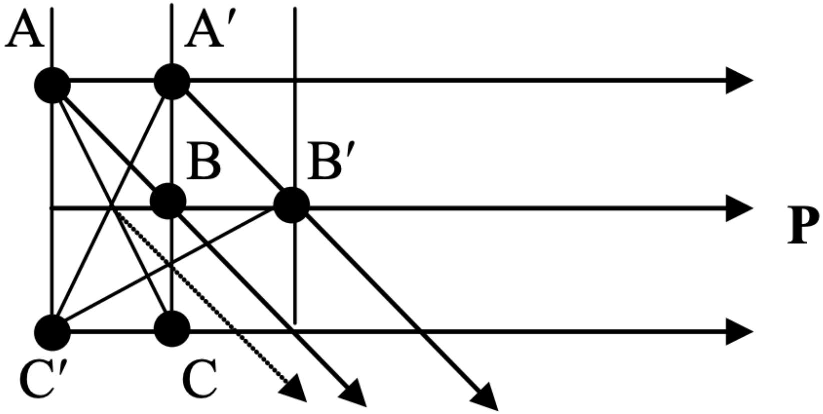
Exercise 3: Triangles AFG and BDE are in perspective from C.
Let R = AF$\cap$BD, S = AG$\cap$BE and T = FG$\cap$DE. Then R, S, T are collinear. Since AF is parallel to BD and AG is parallel to BE, the points R, S are ideal points and so RS is the ideal line.
Thus T must also be an ideal point. It follows that FG is parallel to DE.

Exercise 4:
AB = $\langle(0, 0, 1)\rangle^\perp$, A′B′ = $\langle(-1, -2, 4)\rangle^\perp$ so
R = $\langle(2, -1, 0)\rangle$.
AC = $\langle(0, -1, 0)\rangle^\perp$, A′C′ = $\langle(1, -5, 3)\rangle^\perp$ so
S = $\langle(-3, 0, 1)\rangle$.
BC = $\langle(1, 0, 0)\rangle^\perp$, B′C′ = $\langle(6, -2, -3)\rangle^\perp$ so
T = $\langle(0, 3, -2)\rangle$.
RS = $\langle(-1, -2, -3)\rangle^\perp = \langle(1, 2, 3)\rangle^\perp$,
ST = $\langle(-3, -6, -9)\rangle^\perp$ = RS.

[We could have simply observed that:
$$3(2, -1, 0) + 2(-3, 0, 1) + (0, 3, -2) = (0, 0, 0)$$
and so R, S, T are collinear.]

Exercise 5:

(i)



AA′, BB′, CC′ are all horizontal. So **P is the ideal point on the horizontal lines**.

AC∩A′C′ = (½, 1). AB∩A′B′ is the ideal point on the lines with slope –1.

So h is the line through (½, 1) with slope –1, ie the line $x + y = 3/2$.

(ii) (1, 4, 7) × (2, 3, 5) = (–1, 9, –5) so:

$$AA' = \langle(-1, 9, -5)\rangle^{\perp}.$$

(2, –1, 3) × (1, 1, –1) = (–2, 5, 3) so:

$$BB' = \langle(-2, 5, 3)\rangle^{\perp}.$$

$P = \langle(-1, 9, -5) \times \langle(-2, 5, 3)\rangle^{\perp}$

$= \langle(52, 13, 13)\rangle = \langle(4, 1, 1)\rangle$.

$CC' = \langle(3, 5, 3) \times (1, -4, -2)\rangle^{\perp} = \langle(2, 9, -17)\rangle^{\perp}$.

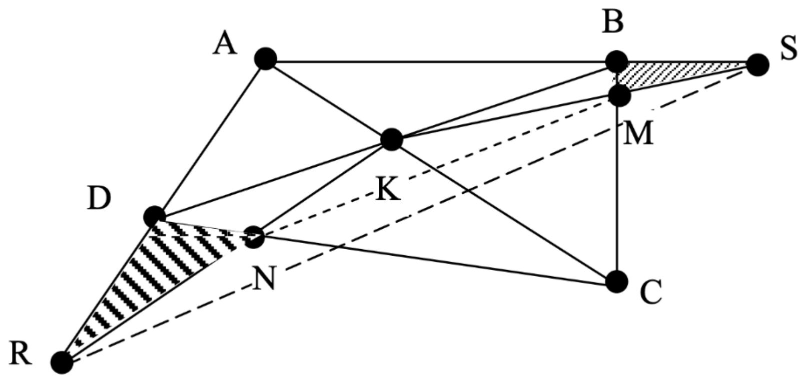
Since (2, 9, –17).(4, 1, 1) = 0, P lies on CC′.

Thus the triangles are in perspective from **P = ⟨(4, 1, 1)⟩**.

$AB = \langle(19, 11, -9)\rangle^{\perp}$ and $A'B' = \langle(-8, 7, -1)\rangle^{\perp}$.
So $AB \cap A'B' = \langle(52, 91, 221)\rangle = \langle(4, 7, 17)\rangle$.
$BC = \langle(-18, 3, 13)\rangle^{\perp}$ and $B'C' = \langle(-6, 1, -5)\rangle^{\perp}$ so
$BC \cap B'C' = \langle(-28, -168, 0)\rangle = \langle(1, 6, 0)\rangle$.
Thus $h = \langle(4, 7, 17) \times (1, 6, 0)\rangle^{\perp}$
$= \langle(-102, 17, 17)\rangle^{\perp} = \boldsymbol{\langle(-6, 1, 1)\rangle^{\perp}}$.
We check that $AC \cap A'C'$ lies on h.
$AC = \langle(-23, 18, -7)\rangle^{\perp}$ and $A'C' = \langle(14, 9, -11)\rangle^{\perp}$ so
$AC \cap A'C' = \langle(-135, -351, -459)\rangle^{\perp} = \langle(5, 13, 17)\rangle^{\perp}$.
Since $(5, 13, 17).(-6, 1, 1) = 0$ then $AC \cap A'C'$ lies on h.

Exercise 6: (i) $\mathbf{p} - x\mathbf{b} = (\mathbf{a} + \mathbf{p}) - (\mathbf{a} + x\mathbf{b}) \in PB \cap QX$ so
$R = \langle \mathbf{p} - x\mathbf{b} \rangle$.
(ii) $(\mathbf{a} + \mathbf{p}) + y\mathbf{b} = (\mathbf{a} + y\mathbf{b}) + \mathbf{p} \in QB \cap YP = S$ so
$S = \langle \mathbf{a} + \mathbf{p} + y\mathbf{b} \rangle$.
(iii) Since R, S, Z are collinear there exist scalars α, β, γ, not all zero, such that:
$\alpha(\mathbf{p} - x\mathbf{b}) + \beta(\mathbf{a} + \mathbf{p} + y\mathbf{b}) + \gamma(\mathbf{a} + z\mathbf{b}) = \mathbf{0}$.
Since **p**, **a**, **b** are linearly independent we have:
$\alpha + \beta = 0$, $\beta + \gamma = 0$, $-\alpha x + \beta y + \gamma z = 0$.
Thus $\beta(x + y - z) = 0$.
Now $\beta \neq 0$ so $x + y = z$.

Exercise 7:



Triangles DNR and BMS are in perspective from the line AKC and hence they are in perspective from a point, by the converse of Desargues' Theorem. Hence the lines BD, MN and RS are concurrent.